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ALLOWANCE FOR TEMPERATURE SENSITIVITY IN THE PROBLEM OF DIAGNOSTICS OF THERMOELASTIC MEDIA

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The problem of determining thermomechanical characteristics of a medium, which are functions of spatial variables and temperature, from the values of characteristics of thermoelastic processes measured at the half-space boundary is considered. An approach to solving the problem, based on the use of the method of perturbations, is proposed.

Key words: *thermoelastic medium, unsteady processes, method of perturbations, diagnostics.*

Improvement of elements of many devices (from vapor and gas turbines and internal combustion engines to space engineering) designed for operation under conditions of increasing temperatures and pressures makes it necessary to study the temperature dependence of characteristics of complex metallic alloys and other structural materials. Experimental studies of temperature sensitivity involve significant technical difficulties [1], especially in considering advanced composite materials, which are inhomogeneous and anisotropic.

Possessing a number of advantages important in engineering (having better exploitation qualities than any of their components), composite materials have also significant drawbacks. For instance, there is often disagreement between the physicomechanical and chemical properties of composites, which lead to specific types of failure (stratification, local disruptions, violation of adhesion, etc.) [2]. Therefore, the problem of nondestructive control of the quality of articles becomes urgent both immediately after article fabrication and in the course of its exploitation, which involves accumulation of microfailures.

The problem of diagnostics is understood as the problem of determining material characteristics on the basis of experimental information on physical fields arising in the body under the influence of specially chosen external actions [3].

In the present work, we study the problem of determining the temperature dependence of rigidity characteristics, density, specific heat, and heat-transfer coefficients for a weakly inhomogeneous and anisotropic thermoelastic medium on the basis of the parameters of unsteady processes in the body, which are assumed to be known on the body surface. The problem under study is an inverse problem of mathematical physics [4]. The main difference of this problem from the formulations of inverse problems considered previously in [5–13] is determining the characteristics of the medium, which are functions of not only spatial variables but also of temperature (temperature sensitivity). Introduction of the small parameter is related to the nonuniform temperature dependence of the characteristics of the medium. The use of power decompositions of the characteristics of the medium in terms of relative temperature is similar to the use of the method of perturbations.

1. Propagation of unsteady thermoelastic processes in an inhomogeneous anisotropic half-space $R^3_+ = \{(x_1, x_2, x_3) \mid x_3 \geqslant 0\}$ is described by the equations [14]

$$C_v\dot{\theta} - (K_{ij}\theta_{,i})_{,j} = f_0, \qquad \rho\ddot{u}_i - (C_{ijkl}u_{k,l} - \beta_{ij}\theta)_{,j} = f_i \quad (i, j, k, l = 1, 2, 3), \tag{1.1}$$

which are closed by the initial and boundary conditions

$$\begin{gathered} \theta(\boldsymbol{x}, 0) = \varphi_0(\boldsymbol{x}), \quad u_i(\boldsymbol{x}, 0) = \varphi_i(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = \psi_i(\boldsymbol{x}), \\ K_{i3}\theta_{,i}(x_1, x_2, 0, t) = p_0(x_1, x_2, t), \qquad \{C_{i3kl}u_{k,l} - \beta_{i3}\theta\}(x_1, x_2, 0, t) = p_i(x_1, x_2, t). \end{gathered} \tag{1.2}$$

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Here, the temperature and components of the displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)$ depend on the spatial variables $\boldsymbol{x} = (x_1, x_2, x_3)$ and time t, and the specific heat C_v, density ρ, and components of the tensors of thermal conductivity K_{ij}, volume temperature expansion $\beta_{ij} = C_{ijkm} \int_0^\theta \alpha_{km}\, d\theta$ (α_{km} are the coefficients of linear temperature expansion), and rigidity C_{ijkl} are assumed to be functions of $\boldsymbol{x}$ and temperature θ. The dots indicate derivatives in time, and the subscript after the comma corresponds to derivatives with respect to the corresponding coordinate. Summation is performed over repeated subscripts (unless otherwise indicated). The braces contain functions (expressions) with identical arguments (subscripts).

The problem of diagnostics considered in the present work implies determining the quantities $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}(\boldsymbol{x}, \theta)$ from several types of problems of the form (1.1), (1.2) for N types of heat-force loading [after substitution of $u_i^n \to u_i$, $\{\varphi_r, \psi_i, p_i, f_i\}^n \to \{\varphi_r, \psi_i, p_i, f_i\}$ ($i = 0, 1, 2, 3$; $r = 1, 2, 3$; $n = 1, 2, \ldots, N$) into (1.1) and (1.2)], based on the additional information

$$\begin{gathered}\theta^n(x_1, x_2, 0, t) = \chi^n(x_1, x_2, t), \qquad u_i^n(x_1, x_2, 0, t) = \chi_i^n(x_1, x_2, t),\\ \{K_{i3}, \beta_{i3}, C_{i3kl}\}(x_1, x_2, 0, \theta^n) = \{K_{i3}, \beta_{i3}, C_{i3kl}\}^0 \qquad (i, k, l = 1, 2, 3),\end{gathered} \tag{1.3}$$

which is assumed to be obtained by measurements. The number N corresponding to the number of tests with different types of heat-force loading depends on the type of anisotropy of the medium under study (number of sought functions). In what follows, we assume that the medium considered is weakly inhomogeneous and anisotropic, i.e., the quantities $|\rho - \rho^0|/\rho^0$, $|C_v - C_v^0|/C_v^0$, $|K_{ij} - K_{ij}^0|/K^0$, and $|C_{ijkl} - C_{ijkl}^0|/\lambda^0$ have the order of smallness $O(\varepsilon)$ ($0 < \varepsilon \ll 1$); the zero superscript indicates the characteristics of a certain homogeneous isotropic temperature-insensitive control medium; hence, we have $K_{ij}^0 = K^0\delta_{ij}$, $\beta_{ij}^0 = \beta^0\delta_{ij}$, and $C_{ijkl}^0 = \lambda^0\delta_{ij}\delta_{kl} + \mu^0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Here λ^0 and μ^0 are the Lamé coefficients, δ_{ij} is the Kronecker delta, and ρ^0, C_v^0, K^0, λ^0, and μ^0 are constants. Weak inhomogeneity and anisotropy of thermomechanical properties of the material may be caused, for instance, by technological actions. Under irradiation of metals, Young's modulus of copper changes by 10–15%, Poisson's ratio remains almost unchanged, whereas the yield strength increases severalfold. Determining, within the framework of the diagnostics problem, regions with deviation of thermomechanical characteristics of the material and, thus, identifying zones with the critical level of radiative failures, we can evaluate the strength margin of an article.

Let us compare a thermoelastic process $\{\theta, u\}^n(\boldsymbol{x}, t)$ to a similarly initiated process $\{\theta, u\}^{0n}(\boldsymbol{x}, t)$ proceeding in the homogeneous isotropic temperature-insensitive control medium. After the substitution $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^0 \to \{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}(\boldsymbol{x}, \theta)$, the quantities $\{\theta, u\}^{0n}(\boldsymbol{x}, t)$ are described by relations (1.1) and (1.2). We assume that temperature sensitivity, weak inhomogeneity, and anisotropy of the examined medium have a small effect on the quantitative characteristics of the processes excited in the medium. Thus, on the half-space surface, we have

$$\theta^n(x_1, x_2, 0, t) = \theta^{0n}(x_1, x_2, 0, t) + \varepsilon\theta^{1n}(x_1, x_2, 0, t) = \chi^{0n}(x_1, x_2, t) + \varepsilon\chi^{1n}(x_1, x_2, t),$$

$$u_i^n(x_1, x_2, 0, t) = u_i^{0n}(x_1, x_2, 0, t) + \varepsilon u_i^{1n}(x_1, x_2, 0, t) = \chi_i^{0n}(x_1, x_2, t) + \chi_i^{1n}(x_1, x_2, t),$$

and

$$\|\theta^{0n}(x_1, x_2, 0, t)\|_{C^2} \sim O(\|\theta^{1n}(x_1, x_2, 0, t)\|_{C^2}),$$

$$\|u_i^{0n}(x_1, x_2, 0, t)\|_{C^2} \sim O(\|u_i^{1n}(x_1, x_2, 0, t)\|_{C^2}).$$

We assume that the sought characteristics of the medium under study can be represented as a converging power series in ε and $\theta^{0n}(\boldsymbol{x}, t)$ with coefficients depending on the spatial coordinates $\boldsymbol{x}$:

$$\begin{aligned}\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}(\boldsymbol{x}) &= \{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^0\\ &+ \varepsilon\sum_{s=0}^{\infty} \varepsilon^s(\theta^{0n})^s\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^{s+1}(\boldsymbol{x}),\end{aligned} \tag{1.4}$$

and the characteristics of thermoelastic processes in the medium are analytical functions of the small parameter ε:

$$\{\theta, u\}^n(\boldsymbol{x}, t) = \{\theta, u\}^{0n}(\boldsymbol{x}, t) + \varepsilon\sum_{s=0}^{\infty} \varepsilon^s\{\theta, u\}^{sn}(\boldsymbol{x}, t).$$

In accordance with the method of perturbations, the assumptions adopted allow us to pass from relations (1.1)–(1.3) to the equations

$$
\begin{gathered}
C_v^0 \dot\theta^{0n} - (K_{ij}^0 \theta_{,i}^{0n})_{,j} = f_0^n \qquad (i,j,k,l = 1,2,3;\ n = 1,\ldots,N),\\
\rho^0 \ddot u_i^{0n} - (C_{ijkl}^0 u_{k,l}^{0n} - \beta_{ij}^0 \theta^{0n})_{,j} = f_i^n, \quad \varepsilon(C_v^0 \dot\theta^{1n} - (K_{ij}^0 \theta_{,i}^{1n})_{,j}) = \varepsilon(-C_v^1 \dot\theta^{0n} + (K_{ij}^1 \theta_{,i}^{0n})_{,j}),\\
\varepsilon(\rho^0 \ddot u_i^{1n} - (C_{ijkl}^0 u_{k,l}^{1n} - \beta_{ij}^0 \theta^{1n})_{,j}) = \varepsilon(-\rho^1 \ddot u_i^{0n} + (C_{ijkl}^1 u_{k,l}^{0n} - \beta_{ij}^1 \theta^{0n})_{,j}),\\
\varepsilon^2(C_v^0 \dot\theta^{2n} - (K_{ij}^0 \theta_{,i}^{2n})_{,j}) = \varepsilon^2(-C_v^2 \theta^{0n} \dot\theta^{0n} + (K_{ij}^2 \theta^{0n} \theta_{,i}^{0n})_{,j} - C_v^1 \dot\theta^{1n} + (K_{ij}^1 \theta_{,i}^{0n})_{,j}),\\
\varepsilon^2(\rho^0 \ddot u_i^{2n} - (C_{ijkl}^0 u_{k,l}^{2n} - \beta_{ij}^0 \theta^{2n})_{,j}) = \varepsilon^2(-\rho^2 \theta^{0n} \ddot u_i^{0n} + (C_{ijkl}^2 \theta^{0n} u_{k,l}^{0n} - \beta_{ij}^2 \theta^{0n} \theta^{0n})_{,j}\\
- \rho^1 \ddot u_i^{1n} + (C_{ijkl}^1 u_{k,l}^{1n} - \beta_{ij}^1 \theta^{1n})_{,j}),\\
\varepsilon^m(C_v^0 \dot\theta^{mn} - (K_{ij}^0 \theta_{,i}^{mn})_{,j}) = \varepsilon^m(-C_v^m \theta^{0n} \dot\theta^{0n} + (K_{ij}^m \theta^{0n} \theta_{,i}^{0n})_{,j} - C_v^m (\theta^{0n})^{m-1} \dot\theta^{1n}\\
+ (K_{ij}^m (\theta^{0n})^{m-1} \theta_{,i}^{1n})_{,j} - \ldots - C_v^1 \dot\theta^{(m-1)n} + (K_{ij}^1 \theta_{,i}^{(m-1)n})_{,j}),\\
\varepsilon^m(\rho^0 \ddot u_i^{mn} - (C_{ijkl}^0 u_{k,l}^{mn} - \beta_{ij}^0 \theta^{mn})_{,j}) = \varepsilon^m(-\rho^m \theta^{0n} \ddot u_i^{0n} + (C_{ijkl}^m \theta^{0n} u_{k,l}^{0n} - \beta_{ij}^m \theta^{0n} \theta^{0n})_{,j}\\
- \rho^1 \ddot u_i^{1n} + (C_{ijkl}^1 u_{k,l}^{(m-1)n} - \beta_{ij}^1 \theta^{(m-1)n})_{,j})
\end{gathered}
\tag{1.5}
$$

closed by the initial and boundary conditions

$$
\begin{gathered}
\theta^{mn}(\boldsymbol{x},0) = \delta_{0m} \varphi_0^n(\boldsymbol{x}), \quad u_i^{mn}(\boldsymbol{x},0) = \delta_{0m} \varphi_i^n(\boldsymbol{x}), \quad \dot u_i^{mn}(\boldsymbol{x},0) = \delta_{0m} \psi_i^n(\boldsymbol{x}),\\
\sum_{s=0}^{m} K_{i3}^{m-s} \theta_{,i}^s(x_1,x_2,0,t) = \delta_{0m} p_0^n(x_1,x_2,t),\\
\sum_{s=0}^{m} \{C_{i3kl}^{m-s} u_{k,l}^s - \beta_{i3}^{m-s} \theta^s\}(x_1,x_2,0,t) = \delta_{0m} p_i^n(x_1,x_2,t);
\end{gathered}
\tag{1.6}
$$

$$
\begin{gathered}
\theta^{mn}(x_1,x_2,0,t) = \delta_{0m} \chi^{0n}(x_1,x_2,t) + \delta_{1m} \chi^{1n}(x_1,x_2,t),\\
u_i^{mn}(x_1,x_2,0,t) = \delta_{0m} \chi_i^{0n}(x_1,x_2,t) + \delta_{1m} \chi_i^{1n}(x_1,x_2,t),\\
\{K_{i3}, \beta_{i3}, C_{i3kl}\}^m(x_1,x_2,0) = \delta_{0m} \{K_{i3}, \beta_{i3}, C_{i3kl}\}^0 \qquad (i,k,l = 1,2,3).
\end{gathered}
\tag{1.7}
$$

In what follows, the characteristics of the control medium $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^0$ are assumed to be known, i.e., the problem of diagnostics implies refinement of the properties of the medium under study. Note, the initial problem of determining C_v, ρ, K_{ij}, β_{ij}, C_{ijkl}, θ^n, and u_i^n from N relations of the form (1.1)–(1.3) is nonlinear, since they contain products of the sought functions (characteristics of the medium and thermoelastic processes). In addition, the dependence of the characteristics of the medium on one of the sought functions of temperature is assumed to be nonlinear. In this sense, the transition from (1.1)–(1.3) to (1.5) is similar to the linearization procedure based on the method of perturbations and used to solve nonlinear problems of thermoelasticity of inhomogeneous bodies [4].

The structure of relations (1.5)–(1.7) allows us to find consecutively the expansion coefficients in the temperature dependence of the characteristics of the medium $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^m$ beginning from $m = 1$.

We give the algorithm for calculating the quantities $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^m(\boldsymbol{x})$ and $\{\theta, u\}^{mn}(\boldsymbol{x}, t)$. Note, the first matrix equation (for $m = 0$) in (1.4) does not contain the unknowns $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^m$ $(m = 1, 2, \ldots)$ and, together with conditions (1.6) considered for $m = 0$ forms a series of initial-boundary problems of the second kind [with respect to the functions $\{\theta, u\}^{0n}(\boldsymbol{x}, t)$, $n = 1, 2, \ldots, N$] that describe propagation of thermoelastic waves in a homogeneous isotropic temperature-insensitive half-space. In what follows, we assume that the solutions of these problems are known and have the form $\{\theta, u_i\}^{0n}(\boldsymbol{x}, t) = \exp(-a_n t)\{g_0, g_i\}^n(\boldsymbol{x})$, $a_n > 0$ (no summation in terms of n is performed), which imposes restrictions on the functions $\{f_0, f_i, \varphi_0, \varphi_i, \psi_i, p_0, p_i\}^n$, i.e., on the conditions of initiation of thermoelastic processes in the medium under study.

Note, the particular form of the functions $\{f_0, f_i, \varphi_0, \varphi_i, \psi_i, p_0, p_i\}^n$ can be obtained by the direct substitution of $\{\theta, u_i\}^{0n}(\boldsymbol{x}, t)$ into (1.1)–(1.3) after the replacement $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^0 \to \{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}(\boldsymbol{x}, \theta)$. The monotonic change in temperature in the control medium corresponds to the physical assumptions made previously.

2. In accordance with (1.5)–(1.7), for the functions $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}, \theta^n, u^n\}^1$, we have

$$C_v^0 \dot{\theta}^{1n} - (K_{ij}^0 \theta_{,i}^{1n})_{,j} = -C_v^1 \dot{\theta}^{0n} + (K_{ij}^1 \theta_{,i}^{0n})_{,j}; \tag{2.1}$$

$$\rho^0 \ddot{u}_i^{1n} - (C_{ijkl}^0 u_{k,l}^{1n} - \beta_{ij}^0 \theta^{1n})_{,j} = -\rho^1 \ddot{u}_i^{0n} + (C_{ijkl}^1 u_{k,l}^{0n} - \beta_{ij}^1 \theta^{0n})_{,j},$$

$$\theta^{mn}(\boldsymbol{x}, 0) = 0, \quad u_i^{mn}(\boldsymbol{x}, 0) = 0, \quad \dot{u}_i^{mn}(\boldsymbol{x}, 0) = 0, \quad \{K_{i3}^0 \theta_{,i}^{1n} + K_{i3}^1 \theta_{,i}^{0n}\}(x_1, x_2, 0, t) = 0, \tag{2.2}$$

$$\{C_{i3kl}^0 u_{k,l}^{1n} - \beta_{i3}^0 \theta^{1n} + C_{i3kl}^1 u_{k,l}^{0n} - \beta_{i3}^1 \theta^0\}(x_1, x_2, 0, t) = 0,$$

$$\theta^{1n}(x_1, x_2, 0, t) = \chi^{1n}(x_1, x_2, t), \qquad u_i^{1n}(x_1, x_2, 0, t) = \chi_i^{1n}(x_1, x_2, t),$$

$$\{K_{i3}, \beta_{i3}, C_{i3kl}\}^1(x_1, x_2, 0) = 0 \qquad (i, k, l = 1, 2, 3).$$

The problem of determining these functions is similar to the linearized problem of diagnostics [10–12]; therefore, we give only the schematic of its solution. The solution of the problem is divided into two stages: 1) determination of $\{\theta, u_i\}^{1n}(\boldsymbol{x}, t)$ $(n = 1, N)$; 2) reconstruction of $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^1(\boldsymbol{x})$ from the right sides of the matrix equations (2.1) and (2.2).

Stage 1. We apply the operator $L = \partial_t + a_n I$ (I is the unit operator) to Eqs. (2.1) and (2.2); after that, for each fixed n with respect to new unknowns $T = L\theta^{1n}$ and $v_i = Lu_i^{1n}$, we obtain

$$C_v^0 \dot{T} - (K_{ij}^0 T_{,i})_{,j} = 0; \tag{2.3}$$

$$\rho^0 \ddot{v}_i - (C_{ijkl}^0 v_{k,l} - \beta_{ij}^0 T)_{,j} = 0; \tag{2.4}$$

$$v_i(\boldsymbol{x}, 0) = 0; \tag{2.5}$$

$$K_{i3}^0 T_{,i}(x_1, x_2, 0, t) = 0, \qquad T(x_1, x_2, 0, t) = L\chi^{1n}(x_1, x_2, t); \tag{2.6}$$

$$C_{i3kl}^0 v_{k,l}(x_1, x_2, 0, t) = 0, \quad v_i(x_1, x_2, 0, t) = L\chi_i^{1n}(x_1, x_2, t) \quad (i = 1, 2, 3). \tag{2.7}$$

Equations (2.3) and (2.4) are similar to the equations of thermoelasticity for a homogeneous isotropic medium, since the quantities $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkm}\}^0$ entering into these equations as coefficients correspond to a homogeneous isotropic medium. Nevertheless, in our case, they are used to find the auxiliary functions T and v_i. Since $K_{ij}^0 = K^0 \delta_{ij}$, then Eq. (2.1) with the boundary conditions (2.6) is the Cauchy problem with data on the non-three-dimensional manifold [4].

The boundary conditions (2.7) with allowance for (2.4) make it possible to find the functions $\{U, U_{,3}, \boldsymbol{W}, \boldsymbol{W}_{,3}\}$ for $x_3 = 0$ [$U = \operatorname{div} \boldsymbol{v}$ and $\boldsymbol{W} = (W_1, W_2, W_3) = \operatorname{rot} \boldsymbol{v}$]. It follows from condition (2.5) that the initial conditions for these functions are homogeneous: $U(\boldsymbol{x}, 0) = 0$ and $\boldsymbol{W}(\boldsymbol{x}, 0) = 0$. Applying the operators div and rot to Eqs. (2.4), we obtain

$$\rho^0 \ddot{U} - (\lambda^0 + 2\mu^0)\Delta U = -\beta^0 \Delta T, \qquad \rho^0 \ddot{\boldsymbol{W}} - \mu^0 \Delta \boldsymbol{W} = 0.$$

Thus, for U and three components of the vector-function $\boldsymbol{W}$ (only two of which are independent), we obtain a wave equation, a homogeneous initial condition, and two boundary conditions for $x_3 = 0$, which form a nonhyperbolic Cauchy problem (Cauchy problem with data on the non-three-dimensional manifold) for the wave equation. This problem, as problem (2.3), (2.6), is classically ill-posed in the class of functions $C^n(R_+^3 \times R_+)$: its solution exists for not all Cauchy data (for $x_3 = 0$) of this class. This makes the problem of diagnostics as a whole ill-posed too. It should be noted that the problems considered are classically (according to Hadamard) well-posed in the class of analytical functions. In particular, if the boundary values of the sought functions admit expansions into a series in eigenfunctions of the Laplace operator as representations, then the solution can be obtained by the Fourier method of division of variables. Finding U, W_1, and W_2, we can reconstruct $\boldsymbol{v} = (v_1, v_2, v_3)$; then, from the equations

$$\dot{\theta}^{1n} + a_n \theta^{1n} = T, \qquad \dot{u}_i^{1n} + a_n u_i^{1n} = v_i$$

using the homogeneous initial conditions, we can determine $\{\theta, u_i\}^{1n}(\boldsymbol{x}, t)$. The first-stage calculations are performed N times.

Stage 2. Knowing θ^{1n} and u_i^{1n}, we can find the right sides of Eqs. (2.1) and (2.2), which, in accordance with the adopted assumption, have the form $\exp(-a_n t)\{F_0^{1n}, F_1^{1n}, F_2^{1n}, F_3^{1n}\}(\boldsymbol{x})$. Thus, the second stage implies determination of $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^1(\boldsymbol{x})$ from the equations

$$a_n C_v^1 g_0^n + (K_{ij}^1 g_{0,i}^n)_{,j} = F_0^{1n}, \quad -(a_n)^2 \rho^1 g_i^n + (C_{ijkl}^1 g_{k,l}^n - \beta_{ij}^1 g_0^n)_{,j} = F_i^{1n} \quad (i = 1, 2, 3) \tag{2.8}$$

and homogeneous boundary conditions with respect to $\{K_{i3}, \beta_{i3}, C_{i3kl}\}^1$ from (2.2). The number N (number of different test regimes) is chosen such that the number of independent scalar equations of the form (2.8) is equal to the number of independent unknowns $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^1(\boldsymbol{x})$.

In the case of the general-form anisotropy, the tensor of rigidity coefficients C_{ijkl} contains 21 independent components, and the tensors of thermal conductivity coefficients K_{ij} and volume temperature expansion coefficients β_{ij} contain six independent components each. In this case, the total number of unknowns is 35. The high order hinders the solution of the system of the form (2.8). Nevertheless, consideration of special types of anisotropy and additional functional relations between thermomechanical characteristics of the material and also an appropriate choice of the functions $\{g_0, g_1, g_2, g_3\}^n$ (test conditions) allow one sometimes to reduce the order and simplify the solution.

3. To determine the quantities $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}, \theta^n, u^n\}^2$ from Eqs. (1.5)–(1.7), we have

$$\begin{gathered} C_v^0 \dot\theta^{2n} - (K_{ij}^0 \theta_{,i}^{2n})_{,j} = -C_v^2 \theta^{0n} \dot\theta^{0n} + (K_{ij}^2 \theta^{0n} \theta_{,i}^{0n})_{,j} + [-C_v^1 \dot\theta^{1n} + (K_{ij}^1 \theta_{,i}^{1n})_{,j}], \\ \rho^0 \ddot u_i^{2n} - (C_{ijkl}^0 u_{k,l}^{2n} - \beta_{ij}^0 \theta^{2n})_{,j} = -\rho^2 \theta^{0n} \ddot u_i^{0n} + (C_{ijkl}^2 \theta^{0n} u_{k,l}^{0n} - \beta_{ij}^2 \theta^{0n} \theta^{0n})_{,j} \\ + [-\rho^1 \ddot u_i^{1n} + (C_{ijkl}^1 u_{k,l}^{1n} - \beta_{ij}^1 \theta^{1n})_{,j}]. \end{gathered} \tag{3.1}$$

The initial conditions are

$$\theta^{2n}(\boldsymbol{x}, 0) = 0, \quad u_i^{2n}(\boldsymbol{x}, 0) = 0, \quad \dot u_i^{2n}(\boldsymbol{x}, 0) = 0, \tag{3.2}$$

and the boundary conditions are

$$\begin{gathered} \{K_{i3}^2 \theta_{,i}^0 + [K_{i3}^1 \theta_{,i}^1] + K_{i3}^0 \theta_{,i}^2\}(x_1, x_2, 0, t) = 0, \\ \{C_{i3kl}^2 u_{k,l}^0 + [C_{i3kl}^1 u_{k,l}^1] + C_{i3kl}^0 u_{k,l}^2\}(x_1, x_2, 0, t) = 0; \end{gathered} \tag{3.3}$$

$$\theta^{2n}(x_1, x_2, 0, t) = 0, \qquad u_i^{2n}(x_1, x_2, 0, t) = 0 \qquad (i = 1, 2, 3); \tag{3.4}$$

$$\{K_{i3}, \beta_{i3}, C_{i3kl}\}^2(x_1, x_2, 0) = 0 \qquad (i, k, l = 1, 2, 3). \tag{3.5}$$

In relations (3.1)–(3.5), the square brackets contain expressions with already known (see Sec. 2) functions. The sought solution $\{\theta^n, \boldsymbol{u}^n\}^2(\boldsymbol{x}, t)$ is represented in the form $\{\theta^n, \boldsymbol{u}^n\}^2 = \{\theta^n, \boldsymbol{u}^n\}^{2*} + \{\theta^n, \boldsymbol{u}^n\}^{2**}$, where $\{\theta^n, \boldsymbol{u}^n\}^{2**}$ satisfies Eqs. (3.1)–(3.3) whose right sides are the expressions in square brackets. The problem of determining $\{\theta^n, \boldsymbol{u}^n\}^{2**}$ consists of several initial-boundary problems of dynamic thermoelasticity of the second kind, which describe propagation, in the half-space, of thermoelastic waves initiated by the action of mass forces, heat sources, and heat-force loading on the boundary [14]. After finding $\{\theta^n, u^n\}^{2**}$, the problem of determining $\{\theta^n, \boldsymbol{u}^n\}^{2*}(\boldsymbol{x}, t)$, and then $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^2(\boldsymbol{x})$ from Eqs. (3.1)–(3.4), which do not contain bracketed expressions, is similar to problem (2.1), (2.2). In this case, however, the operator L has the form $L = \partial_t + 2a_n I$.

Thus, it is possible to obtain consecutively all terms of expansions of the characteristics of thermoelastic processes $\{\theta^n, \boldsymbol{u}^n\}^m(\boldsymbol{x}, t)$ used for diagnostics and, which is more important within the framework of the present problem, coefficients of expansion of the characteristics of the medium under study $\{C_v, \rho, K_{ij}, \beta_{ij}, C_{ijkl}\}^m(\boldsymbol{x})$ into a series in powers of temperature, which depend on spatial coordinates.

4. As an example, we consider the simplest case where it is known that the examined medium is isotropic, and its characteristics depend only on the coordinate x_3 (distance to the half-space surface). Then, for conditions of initiation of thermoelastic waves independent of x_1 and x_2, the problem of diagnostics becomes two-dimensional: the characteristics of the medium depend on the spatial coordinate x_3 and temperature θ, whereas the characteristics of thermoelastic processes depend on x_3 and time t.

The number of characteristics of the medium to be found decreases to six: $\{C_v, K, \rho, \lambda, \mu, \beta\}(x_3, \theta)$; therefore, we assume that $N = 2$. For $m = 1$, relations (2.1) and (2.2) take the form

$$C_v^0 \dot\theta^{1n} - K^0 \theta^{1n}_{,ii} = -C_v^1 \dot\theta^{0n} + (K^1 \theta^{0n}_{,i})_{,i} \qquad (n = 1, 2,\ k = 1, 2,\ i = 1, 2, 3),$$

$$\rho^0 \ddot u_k^{11} - \mu^0 \Delta u_k^{11} = -\rho^1 \ddot u_k^{01} + (\mu^1 u^{01}_{k,3})_{,3}, \tag{4.1}$$

$$\rho^0 \ddot u_3^{1n} - (\lambda^0 + 2\mu^0)\Delta^0 u_3^{1n} - \beta^0 \theta^{1n}_{,k} = -\rho^1 \ddot u_3^{0n} + (\mu^1 u^{0n}_{3,3})_{,3} - (\beta^1 \theta^{0n})_{,3}.$$

Relations (4.1) are closed by the initial conditions

$$\theta^{1n}(\boldsymbol{x}, 0) = 0, \quad u_k^{11}(\boldsymbol{x}, 0) = 0, \quad u_3^{1n}(\boldsymbol{x}, 0) = 0, \quad \dot u_k^{11}(\boldsymbol{x}, 0) = 0, \quad \dot u_3^{1n}(\boldsymbol{x}, 0) = 0 \tag{4.2}$$

and the boundary conditions

$$\{K^0 \theta^{1n}_{,i} + K^1 \theta^{0n}_{,i}\}(x_1, x_2, 0, t) = 0,$$

$$\{\mu^0(u^{11}_{k,3} + u^{11}_{3,k}) + \mu^1(u^{01}_{k,3} + u^{01}_{3,k})\}(x_1, x_2, 0, t) = 0 \qquad (k = 1, 2),$$

$$\{\lambda^0(u^{1n}_{1,1} + u^{1n}_{2,2}) + (\lambda^0 + 2\mu^0) u^{1n}_{3,3} - \beta^0 \theta^{1n} + \lambda^1(u^{0n}_{1,1} + u^{0n}_{2,2})$$

$$+ (\lambda^1 + 2\mu^1)(u^{0n}_{3,3}) - \beta^1 \theta^{0n}\}(x_1, x_2, 0, t) = 0, \tag{4.3}$$

$$\theta^{1n}(x_1, x_2, 0, t) = \chi^{1n}(x_1, x_2, t), \quad u_k^{11}(x_1, x_2, 0, t) = \chi_k^{11}(x_1, x_2, t) \quad (k = 1, 2),$$

$$u_3^{1n}(x_1, x_2, 0, t) = \chi_3^{1n}(x_1, x_2, t);$$

$$\{K, \lambda, \mu, \beta\}^1(x_3) = 0. \tag{4.4}$$

Thus, it is necessary to perform two "tests": the temperature and three components of the displacement vector are measured in the first test, and only the temperature and the normal component of displacement are measured in the second test.

The specific feature of the case considered is the fact that the assumption on spatial one-dimensionality (dependence on x_3 only) of inhomogeneity distribution makes the problem one-dimensional at the first stage and, thus, significantly simplifies reconstruction of the characteristics of thermoelastic processes used for diagnostics. In this problem the temperature and strain fields are not related; in addition, longitudinal and transverse waves propagate independently. Therefore, there is no need to separate the divergent and rotor components of the displacement vector. This assumption also simplifies the problem at the second stage, since the coefficients of expansions of the characteristics of the medium in powers of θ also depend on x_3 only. In this case, for $\{g_0, g_1, g_2, g_3\}^n = \{g_0, g_1, g_2, g_3\}^n(x_3)$ [for further simplification, we assume that $g_0^1(x_3) = g_0^2(x_3)$, $g_1^1(x_3) = g_2^1(x_3)$, $g_1^2(x_3) = g_2^2(x_3) = 0$], the solution of the problem at the second stage is found from a system of six ordinary linear differential equations with variable coefficients

$$a_n C_v^1 g_0^n + (g^n_{0,3} K^1)_{,3} = F_0^{1n} \qquad (n = 1, 2); \tag{4.5}$$

$$-(a_1)^2 \rho^1 g_k^1 + (\mu^1 g^1_{k,3})_{,3} = F_k^{11} \qquad (k = 1, 2); \tag{4.6}$$

$$-(a_n)^2 \rho^1 g_3^n + ((\lambda^1 + 2\mu^1) g^n_{3,3})_{,3} - (\beta^1 g_0^n)_{,3} = F_3^{1n} \qquad (n = 1, 2). \tag{4.7}$$

It should be noted that system (4.5)–(4.7) is divided into three subsystems. From Eqs. (4.5) for $n = 1$, 2, we find

$$K^1(x_3) = \frac{1}{g^1_{0,3}} \int\limits_0^{x_3} (a_2 F_0^{11}(\xi) - a_1 F_0^{12}(\xi))\, d\xi,$$

$$C_v^1(x_3) = \frac{1}{a_1 g_0^1} \Big(F_0^{11} - \int\limits_0^{x_3} (a_2 F_0^{11}(\xi) - a_1 F_0^{12}(\xi))\, d\xi \Big).$$

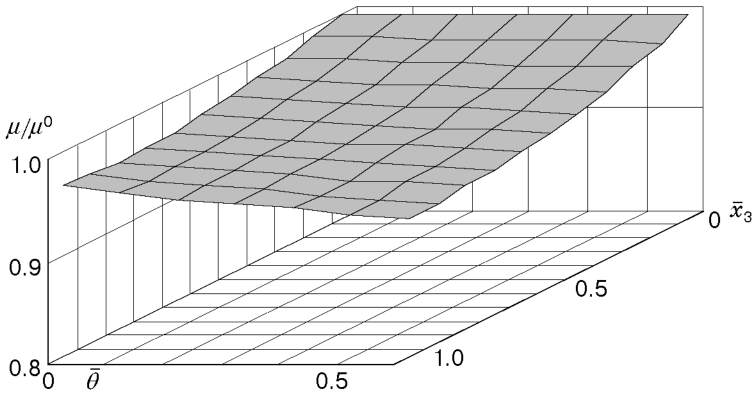

Fig. 1

The equations of system (4.6) for $k = 1$, 2 allow us to find

$$\mu^1(x_3) = \frac{1}{g^1_{1,3}} \int\limits_0^{x_3} ((a_2)^2 F_1^{11}(\xi) - (a_1)^2 F_2^{11}(\xi))\, d\xi,$$

$$\rho^1(x_3) = \frac{1}{(a_1)^2 g_1^1}\Big(F_1^{11} + \int\limits_0^{x_3} ((a_2)^2 F_1^{11}(\xi) - (a_1)^2 F_2^{11}(\xi))\, d\xi\Big).$$

Equations (4.7) for $n = 1$, 2, with allowance for the already found function $\mu^1(x_3)$, yield

$$\lambda^1(x_3) = -2\mu^1(x_3) + \frac{1}{g^1_{3,3} - g^2_{3,3}} \int\limits_0^{x_3} (F_3^{11}(\xi) - F_3^{12}(\xi) + (a_1)^2 \rho^1(\xi) g_3^1(\xi) - (a_2)^2 \rho^1(\xi) g_3^2(\xi))\, d\xi,$$

$$\beta^1(x_3) = \frac{1}{g_0^1}\Big(\int\limits_0^{x_3} (F_3^{11}(\xi) + (a_1)^2 \rho^1(\xi) g_3^1(\xi))\, d\xi - (\lambda^1 + 2\mu^1) g^1_{3,3}\Big).$$

The remaining coefficients of expansions of the characteristics of the examined medium $\{C_v, K, \rho, \lambda, \mu, \beta\}^m(x_3)$ $(m = 1, 2, \ldots)$ into a series in powers of temperature are found in a similar manner.

Let us give some examples of numerical calculations. In relations (4.1)–(4.3), we used the dimensionless variables $\bar C_v = C_v/C_v^0$, $\bar K = K/K^0$, $\bar\rho = \rho/\rho^0$, $\bar\lambda = \lambda/\mu^0$, $\bar\mu = \mu^0$, $\bar\beta = \beta/\beta^0$, $\bar x_3 = x_3/a$, $\bar t = tC_v^0/a$, $\bar\theta = \theta/T$, $\bar u_i = u_i/a$, $\{\bar\theta, \bar u_1, \bar u_2, \bar u_3\}^1(0, \bar t\,) = \{0, 0.3\exp(-\bar t\,) - 0.1\exp(\bar t\,) - 0.2\exp(-2\bar t\,), 0, 0\}$, $\{K, \lambda, \mu, \beta\}(0) = 0$, $\{g_0, g_1, g_2, g_3\}^1(\bar x_3) = \{\exp(-\bar x_3), \cos\bar x_3, \cos\bar x_3, \sin\bar x_3\}$, $a_1 = 1$, and $a_2 = 2$. In expansion (1.4), we took into account three terms of the series, since the terms corresponding to higher powers of θ do not exert any significant effect on the solution in our example. Figure 1 shows the shear modulus μ/μ^0 (μ^0 is the shear modulus of the homogeneous control medium) as a function of the spatial variable $\bar x_3$ and relative temperature $\bar\theta$. The behavior of the shear modulus (as well as other thermomechanical characteristics) with increasing temperature offers additional information on structural inhomogeneity of the material, which is the ultimate objective of diagnostics.

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